REPRESENTATIONS OF WREATH PRODUCTS OF ALGEBRAS

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Abstract. Filtrations of modules over wreath products of algebras are studied and corresponding multiplicity formulas are given in terms of Littlewood-Richardson coefficients. An example relevant to Jantzen filtrations in Schur algebras is presented.

1. Introduction

Let A be a finite-dimensional algebra over a field k and w be a positive integer such that $w!$ is nonzero in k. It is well-known that the simple modules of the wreath product $A \wr \mathfrak{S}_w$ (which we denote by $A(w)$) can be constructed in a systematic way from the simple modules of A and are naturally labelled by tuples of partitions. This construction still makes sense if one starts with a set of not necessarily simple A-modules.

The aim of this paper is to study how filtrations of A-modules induce filtrations of the $A(w)$-modules constructed from them in this way. We give explicit formulas, in terms of Littlewood-Richardson coefficients, for multiplicities of factors in filtrations. This allows, for example, a nice description of the Ext-quiver of $A(w)$ in terms of the Ext-quiver of A, as well as (in the appropriate context) the calculation of decomposition numbers of $A(w)$ from decomposition numbers of A.

Most of our work is no harder if we replace k by a discrete valuation ring. We need this level of generality for an application in [1]: the determination of Jantzen filtrations of Weyl modules in certain well-behaved blocks of Schur algebras.

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After intorducting some notation in section 2 we introduce the wreath product and the basic construction of modules in section 3. Then the main results on filtrations are obtained in section 4. The next two sections concern the eAe construction and quasihereditary algebras. We end with an important example.

2. Preliminaries

Let w be a positive integer. A composition $\mathsf{w} = (w_1, w_2, \dots)$ of w, denoted as $\mathsf{w} \vDash w$, is a sequence of nonnegative integers which sums to w. If $w_i = 0$ for all $i > r$, we usually write $\mathsf{w} = (w_1, w_2, \dots, w_r)$. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of w, denoted as $\lambda \vdash w$, is a composition of w which is non-increasing. Given a partition λ, not necessarily of w, we write $|\lambda|$ for the sum $\sum_j \lambda_j$.

Let Λ be the set of all partitions, and for any set I let Λ^I be the set of I-tuples $\boldsymbol{\lambda} = (\lambda^i)_{i\in I}$ of partitions and Λ^I_w the set of $\boldsymbol{\lambda}$ such that $\sum_{i\in I} |\lambda^i| = w$. If $\geq$ is a partial order on I, then we define a partial order $\succeq$ on Λ^I_w by $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\mu}$ or

$$\sum_{\substack{\gamma\in I\\ \gamma\geq i}} |\lambda^\gamma| \geq \sum_{\substack{\gamma\in I\\ \gamma\geq i}} |\mu^\gamma|$$

holds for all $i \in I$ and holds with a strict inequality for some $i' \in I$.

Given $\lambda \in \Lambda$ and $\lambda^1, \dots, \lambda^s \in \Lambda$ let $c(\lambda; \lambda^1, \dots, \lambda^s)$ be the associated Littlewood-Richardson coefficient if $|\lambda| = \sum_{i=1}^s |\lambda^i|$ and 0 otherwise (see, e.g., [8, I.9]).

Let $\mathfrak{S}(U)$ be the symmetric group on a finite set U. We write $\mathfrak{S}_w$ for $\mathfrak{S}(\{1, \dots, w\})$. If $\mathsf{w} = (w_1, \dots, w_r) \vDash w$, then there is a Young subgroup

$$\mathfrak{S}_{\mathsf{w}} = \mathfrak{S}(\{1, \dots, w_1\}) \times \mathfrak{S}(\{w_1 + 1, \dots, w_1 + w_2\}) \times \cdots \\ \times \mathfrak{S}(\{\sum_{i=1}^{r-1} w_i + 1, \dots, \sum_{i=1}^{r} w_i\}),$$

which we identify in the obvious way with a subgroup of $\mathfrak{S}_w$.

For any partition λ of w and any commutative ring R let S^λ_R be the associated Specht module of the group algebra $R\mathfrak{S}_w$ (see [5, 8.4]). If $R_1 \to R_2$ is a ring homomorphism, then $R_2 \otimes_{R_1} S^\lambda_{R_1} \cong S^\lambda_{R_2}$.

We make use of the following notations and conventions in this paper:

(1) R denotes either a discrete valuation ring or a field.
(2) w is a fixed positive integer such that $w!$ is invertible in R.
(3) A denotes a unitary R-algebra, finitely generated over R.
(4) By an A-module, we mean a finitely generated left A-module.

We shall also write $\otimes$ in place of $\otimes_R$ and S^λ in place of S^λ_R.

If M is a left A-module then $M^\vee = \mathrm{Hom}_R(M, R)$ is a right A-module with action given by $(\phi a)(m) = \phi(am)$ ($\phi \in M^\vee$, $a \in A$, $m \in M$). We shall denote by nM the direct sum of n copies of an A-module M. If M is an A-module and Γ a set of A-modules, we say that M *is filtered by* Γ if there is a filtration

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{m+1} = 0$$

and a bijection between Γ and the $(m+1)$-element set $\{M_i/M_{i+1} \mid 0 \leq i \leq m\}$ of subquotients such that corresponding modules are isomorphic.

If X is an $R\mathfrak{S}_w$-module and $\mathsf{w} \vDash w$, then by restriction of scalars we obtain an $R\mathfrak{S}_{\mathsf{w}}$-module, denoted as $\mathrm{Res}^w_{\mathsf{w}} X$ or just $\mathrm{Res}_{\mathsf{w}} X$. Similarly, if Y is an $R\mathfrak{S}_{\mathsf{w}}$-module, then the induced module $R\mathfrak{S}_w \otimes_{R\mathfrak{S}_{\mathsf{w}}} Y$ is denoted as $\mathrm{Ind}^w_{\mathsf{w}} Y$ or just $\mathrm{Ind}^w Y$.

Lemma 2.1. *Let* $\mathsf{w} = (w_1, \ldots, w_r) \vDash w$.

(1) *For* $i = 1, \ldots, r$, *let* $\lambda^i \vdash w_i$. *Then*

$$\mathrm{Ind}^w_{\mathsf{w}}(S^{\lambda^1} \otimes \cdots \otimes S^{\lambda^r}) \cong \bigoplus_{\lambda \vdash w} c(\lambda; \lambda^1, \ldots, \lambda^r) S^\lambda.$$

(2) *Let* $\lambda \vdash w$. *Then*

$$\mathrm{Res}^w_{\mathsf{w}} S^\lambda \cong \bigoplus_{\lambda^1 \vdash w_1, \ldots, \lambda^r \vdash w_r} c(\lambda; \lambda^1, \ldots, \lambda^r)(S^{\lambda^1} \otimes \cdots \otimes S^{\lambda^r}).$$

Proof. If R is a field, this is well known. If R is a discrete valuation ring, we note that because $w!$ in nonzero in the quotient field K of R and in the residue field k of R the group algebras $K\mathfrak{S}_v$ and $k\mathfrak{S}_v$ are split-semisimple for any $v \leq w$. Thus every $k\mathfrak{S}_v$-module lifts uniquely to an $R\mathfrak{S}_v$-module free over R; in particular S^λ_k lifts uniquely to S^λ if $\lambda \vdash w$ and $S^{\lambda^i}_k$ lifts uniquely to S^{λ^i} if $\lambda^i \vdash w_i$ ($i = 1, 2, \ldots, r$), and thus the result follows.

3. Wreath products

Let A be an R-algebra. The symmetric group $\mathfrak{S}_w$ acts as algebra automorphisms on $T^w(A)$, the w-th fold tensor power of A, by place permutations:

$$\sigma(a_1 \otimes \cdots \otimes a_w) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(w)}.$$

We define an R-algebra

$$A(w) := T^w(A) \otimes R\mathfrak{S}_w$$

with the twisted multiplication

$$(\alpha \otimes \sigma)(\beta \otimes \tau) = \alpha\sigma(\beta) \otimes \sigma\tau \qquad (\alpha, \beta \in T^w(A); \sigma, \tau \in \mathfrak{S}_w).$$

For example, if A is the group algebra of a group G, then $A(w)$ is isomorphic to the group algebra of the wreath product $G \wr \mathfrak{S}_w$.

If $\mathsf{w} = (w_1, \dots, w_r) \vDash w$, then $T^w(A) \otimes R\mathfrak{S}_{\mathsf{w}}$ is a subalgebra of $A(w)$, isomorphic to $A(w_1)\otimes\cdots\otimes A(w_r)$ (where $A(0) = R$ by convention). We shall denote this subalgebra by $A(\mathsf{w})$. If V is an $A(w)$-module then by restriction of scalars we obtain an $A(\mathsf{w})$-module which we denote by $\mathrm{Res}^{A(w)}_{A(\mathsf{w})} V$ or, by an abuse of notation, $\mathrm{Res}^w_{\mathsf{w}} V$. If W is an $A(\mathsf{w})$-module we shall denote the induced module $A(w) \otimes_{A(\mathsf{w})} W$ by $\mathrm{Ind}^{A(w)}_{A(\mathsf{w})} W$ or $\mathrm{Ind}^w_{\mathsf{w}} W$. As a right $A(\mathsf{w})$-module, $A(w)$ is free with basis $\{1 \otimes \sigma \mid \sigma \in \mathfrak{S}_w/\mathfrak{S}_{\mathsf{w}}\}$; hence $\mathrm{Ind}^w_{\mathsf{w}} W = \bigoplus_{\sigma\in\mathfrak{S}_w/\mathfrak{S}_{\mathsf{w}}} (1 \otimes \sigma) \otimes W$.

We list some well known properties of these restriction and induction functors.

Lemma 3.1. *Let* $\mathsf{w} = (w_1, \dots, w_r) \vDash w$.

(1) *If* V *is an* $A(w)$*-module and* W *an* $A(\mathsf{w})$*-module, then*

$$\mathrm{Hom}_{A(w)}(\mathrm{Ind}^w_{\mathsf{w}} W, V) \cong \mathrm{Hom}_{A(\mathsf{w})}(W, \mathrm{Res}^w_{\mathsf{w}} V).$$

(2) *For* $i = 1, \dots, r$*, let* V_i *be an* $A(w_i)$*-module. For* $0 = i_0 < i_1 < \cdots < i_{l-1} < i_l = r$*, let* $v_j = \sum_{s=i_{j-1}+1}^{i_j} w_s$ *for* $j = 1, 2, \dots, l$*. Then* $\mathsf{v} = (v_1, \dots, v_l) \vDash w$ *and*

$$\mathrm{Ind}^{A(w)}_{A(\mathsf{w})}\left(\bigotimes_{s=1}^{r} V_s\right) = \mathrm{Ind}^{A(w)}_{A(\mathsf{v})}\left(\bigotimes_{j=1}^{r} \mathrm{Ind}^{A(v_j)}\left(\bigotimes_{t=i_{j-1}-1}^{i_j} V_t\right)\right).$$

(3) *Let V_i be an $A(w_i)$-module for each $i = 1, \dots, r$, and let $\pi \in \mathfrak{S}_r$. Then*

$$\operatorname{Ind}_{A(w_1,\dots,w_r)}^{A(w)}(V_1 \otimes \cdots \otimes V_r) \cong \operatorname{Ind}_{A(w_{\pi(1)},\dots,w_{\pi(r)})}^{A(w)}(V_{\pi(1)} \otimes \cdots \otimes V_{\pi(r)}).$$

If V is an $A(w)$-module and X is an $R\mathfrak{S}_w$-module then $V \otimes X$ becomes an $A(w)$-module in the following way:

$$(\alpha \otimes \sigma)(v \otimes x) = (\alpha \otimes \sigma)v \otimes \sigma x \qquad (\alpha \in T^w(A), \sigma \in \mathfrak{S}_w, v \in V, x \in X).$$

We denote this $A(w)$-module by $V \oslash X$.

If A is the group algebra of a group G, then $A(w)$ is isomorphic to the group algebra of the wreath product $G \wr \mathfrak{S}_w$ and X may be viewed as an $A(w)$-module via the natural epimorphism $R(G \wr \mathfrak{S}_w) \to R\mathfrak{S}_w$. In this situation $V \oslash X$ is just the usual inner tensor product of two modules over the group algebra.

Similarly, if $\mathsf{w} \vDash w$, and W is an $A(\mathsf{w})$-module and Y is an $R\mathfrak{S}_{\mathsf{w}}$-module, we get an $A(\mathsf{w})$-module $W \oslash Y$.

We tabulate some properties of this construction:

Lemma 3.2. *Let $\mathsf{w} = (w_1, \dots, w_r) \vDash w$.*

(1) *Suppose that for $i = 1, \dots, r$ we have an $A(w_i)$-module V_i and an $R\mathfrak{S}_{\mathsf{w}_i}$-module X_i. Then we have an isomorphism*

$$(V_1 \oslash X_1) \otimes \cdots \otimes (V_r \oslash X_r) \cong (V_1 \otimes \cdots \otimes V_r) \oslash (X_1 \otimes \cdots \otimes X_r)$$

of $A(\mathsf{w})$-modules.

(2) *Suppose that V is an $A(w)$-module and X is a $R\mathfrak{S}_w$-module. Then*

$$\operatorname{Res}_{A(\mathsf{w})}^{A(w)}(V \oslash X) \cong \operatorname{Res}_{A(\mathsf{w})}^{A(w)} V \oslash \operatorname{Res}_{\mathsf{w}}^{w} X$$

(3) *Suppose that V is an $A(w)$-module and Y is a $R\mathfrak{S}_{\mathsf{w}}$-module. Then*

$$V \oslash (\operatorname{Ind}_{\mathsf{w}}^{w} Y) \cong \operatorname{Ind}_{A(\mathsf{w})}^{A(w)}((\operatorname{Res}_{A(\mathsf{w})}^{A(w)} V) \oslash Y)$$

(4) *Suppose that W is an $A(\mathsf{w})$-module and X is a $R\mathfrak{S}_w$-module. Then*

$$(\operatorname{Ind}_{A(\mathsf{w})}^{A(w)} W) \oslash X \cong \operatorname{Ind}_{A(\mathsf{w})}^{A(w)}(W \oslash \operatorname{Res}_{\mathsf{w}}^{w} X)$$

Proof. The first two isomorphisms are given by the obvious identical maps, and the last two are given by

$$v \oslash (\sigma \otimes y) \mapsto (1 \otimes \sigma) \otimes \big((1 \otimes \sigma^{-1})v \oslash y\big)$$

and

$$((1 \otimes \sigma) \otimes w) \oslash x \mapsto (1 \otimes \sigma) \otimes (w \oslash \sigma^{-1}x)$$

respectively.

If M is an A-module, then its w-th fold tensor power $T^w(M)$ is a $T^w(A)$-module, with tensors acting on tensors component-wise. This action extends to $A(w)$ by letting $\mathfrak{S}_w$ act by place permutations and we call the resulting module $T^{(w)}(M)$.

If $\lambda \vdash w$, we define an $A(w)$-module

$$T^\lambda(M) := T^{(w)}(M) \oslash S^\lambda.$$

Remark.

(1) Since $S^{(w)} = R$ is the trivial representation, we see that $T^{(w)}(M) = T^{(w)}(M) \oslash S^{(w)}$, so that there is no ambiguity in the notation $T^{(w)}(M)$.
(2) We define $T^{\emptyset}(M) = R$ by convention.

We have an analogue of Lemma 2.1:

Lemma 3.3. *Let M be an A-module and let $\mathsf{w} = (w_1, \dots, w_r) \vDash w$.*

(1) *For $i = 1, 2, \dots, r$, let $\lambda^i \vdash \mathsf{w}_i$. Then*

$$\mathrm{Ind}_{A(\mathsf{w})}^{A(w)}(T^{\lambda^1}(M) \otimes \cdots \otimes T^{\lambda^r}(M)) \cong \bigoplus_{\lambda \vdash w} c(\lambda; \lambda^1, \dots, \lambda^r) T^\lambda(M).$$

(2) *Let $\lambda \vdash w$. Then*

$$\mathrm{Res}_{A(\mathsf{w})}^{A(w)}(T^\lambda(M)) \cong \bigoplus_{(\lambda^i \vdash w_i)_i} c(\lambda; \lambda^1, \dots, \lambda^r)(T^{\lambda^1}(M) \otimes \cdots \otimes T^{\lambda^r}(M)).$$

Proof. For part (1), we have, using Lemmas 3.2(1,3) and 2.1(1),

$$\begin{aligned}
\operatorname{Ind}_{A(\mathsf{w})}^{A(w)}\left(\bigotimes_{i=1}^{r} T^{\lambda_i}(M)\right) &\cong \operatorname{Ind}_{A(\mathsf{w})}^{A(w)}\left(\bigotimes_{i=1}^{r}(T^{(\mathsf{w}_i)}(M)\oslash S^{\lambda^i})\right)\\
&\cong \operatorname{Ind}_{A(\mathsf{w})}^{A(w)}\left(\bigotimes_{i=1}^{r} T^{(\mathsf{w}_i)}(M)\oslash \bigotimes_{i=1}^{r} S^{\lambda^i}\right)\\
&\cong \operatorname{Ind}_{A(\mathsf{w})}^{A(w)}\left(\left(\operatorname{Res}_{A(\mathsf{w})}^{A(w)} T^{(w)}(M)\right)\oslash \bigotimes_{i=1}^{r} S^{\lambda^i}\right)\\
&\cong T^{(w)}(M)\oslash \operatorname{Ind}_{\mathsf{w}}^{w}(S^{\lambda^1}\otimes\cdots\otimes S^{\lambda^r})\\
&\cong T^{(w)}(M)\oslash\left(\bigoplus_{\lambda\vdash w} c(\lambda;\lambda^1,\dots,\lambda^r)\,S^{\lambda}\right)\\
&\cong \bigoplus_{\lambda\vdash w} c(\lambda;\lambda^1,\dots,\lambda^r)\,T^{\lambda}(M).
\end{aligned}$$

We can prove part (2) similarly, using Lemma 3.2(1,2) and Theorem 2.1(2).

We now consider radical series, in the case where R is a splitting field.

Lemma 3.4. *Suppose that $R = k$ is a splitting field for A.*

(1) *We have* $\operatorname{rad}(T^w(A)) = \sum_{i=0}^{w-1} T^i(A)\otimes\operatorname{rad}(A)\otimes T^{w-i-1}(A)$.

(2) *If A is semisimple, then so is $A(w)$.*

Proof. Part (1) is a well-known fact (see, e.g., [3, proof of (10.38)]). For part (2), we show that every $A(w)$-module is completely reducible. Let M be an $A(w)$-module and let N be an $A(w)$-submodule of M. Since A is semisimple, so is $T^w(A)$. Thus N has a $T^w(A)$-complement in M; let $\pi : M \to N$ be the projection along this complement and define $\pi' : M \to N$, $\pi'(m) = \frac{1}{w!}\sum_{\sigma\in\mathfrak{S}_w}\sigma\pi\sigma^{-1}(m)$. Then π' is a $A(w)$-homomorphism, and $\pi'|_N = \mathrm{id}_N$, so that $M = N\oplus\ker\pi'$ as $A(w)$-modules.

Lemma 3.5. *Suppose that $R = k$ is a splitting field for A.*

(1) $\operatorname{rad}^n(A(w)) = \operatorname{rad}^n(T^w(A))\otimes k\mathfrak{S}_w$ *for all* $n\in\mathbb{N}$.

(2) *If V is an $A(w)$-module, then for all $n\in\mathbb{N}$,*

$$\operatorname{rad}^n(V) = \operatorname{rad}^n(\operatorname{Res}_{T^w(A)}^{A(w)}(V)).$$

(3) *If V is an $A(w)$-module, and X is a $k\mathfrak{S}_w$-module, then for all $n \in \mathbb{N}$,*

$$\operatorname{rad}^n(V \oslash X) = \operatorname{rad}^n(V) \oslash X.$$

(4) *If $\mathsf{w} \models w$ and W is an $A(\mathsf{w})$-module, then for all $n \in \mathbb{N}$,*

$$\operatorname{rad}^n(A(\mathsf{w})) = \operatorname{rad}^n(T^w(A)) \otimes k\mathfrak{S}_{\mathsf{w}};$$
$$\operatorname{rad}^n(\operatorname{Ind}_{A(\mathsf{w})}^{A(w)} W) = \operatorname{Ind}_{A(\mathsf{w})}^{A(w)}(\operatorname{rad}^n(W)).$$

Proof.

(1) Note that, by Lemma 3.4(1), $\operatorname{rad}(T^w(A))$ is invariant under the action of $\mathfrak{S}_w$, so that $(\operatorname{rad}(T^w(A)) \otimes k\mathfrak{S}_w)^n = \operatorname{rad}^n(T^w(A)) \otimes k\mathfrak{S}_w$. Thus $\operatorname{rad}(T^w(A))\otimes k\mathfrak{S}_w$ is a nilpotent ideal, and the quotient of $A(w)$ by it is isomorphic to $(T^w(A)/\operatorname{rad}(T^w(A)))(w)$, which is semisimple by the previous lemma. Thus $\operatorname{rad}(A(w)) = \operatorname{rad}(T^w(A)) \otimes k\mathfrak{S}_w$. Hence

$$\begin{aligned}\operatorname{rad}^n(A(w)) &= (\operatorname{rad}(T^w(A)) \otimes k\mathfrak{S}_w)^n \\ &= \operatorname{rad}^n(T^w(A)) \otimes k\mathfrak{S}_w.\end{aligned}$$

(2) We have

$$\begin{aligned}\operatorname{rad}^n(V) = \operatorname{rad}^n(A(w))V &= (\operatorname{rad}^n(T^w(A)) \otimes k\mathfrak{S}_w)V \\ &= (\operatorname{rad}^n(T^w(A))V = \operatorname{rad}^n(\operatorname{Res}_{T^w(A)}^{A(w)}(V)).\end{aligned}$$

(3) We have

$$\begin{aligned}\operatorname{rad}^n(V \oslash X) &= \operatorname{rad}^n(A(w))(V \oslash X) \\ &= (\operatorname{rad}^n(T^w(A)) \otimes k\mathfrak{S}_w)(V \oslash X) \\ &= (\operatorname{rad}^n(T^w(A)) \otimes k\mathfrak{S}_w)V \oslash X \\ &= \operatorname{rad}^n(A(w))V \oslash X \\ &= \operatorname{rad}^n(V) \oslash X.\end{aligned}$$

(4) Note that $A(\mathsf{w}) \cong T^w(A)\otimes\mathfrak{S}_\mathsf{w}$, so we show $\mathrm{rad}^n(A(\mathsf{w})) = \mathrm{rad}^n(T^w(A))\otimes k\mathfrak{S}_\mathsf{w}$ using similar argument as part (1). Now,

$$\begin{aligned}\mathrm{rad}^n(A(w)\otimes_{A(\mathsf{w})} W) &= (\mathrm{rad}^n(T^w(A))\otimes k\mathfrak{S}_w)(A(w)\otimes_{A(\mathsf{w})} W)\\ &= A(w)\otimes_{A(\mathsf{w})} \mathrm{rad}^n(T^w(A))W\\ &= A(w)\otimes_{A(\mathsf{w})} \mathrm{rad}^n(W).\end{aligned}$$

We now introduce the key construction of $A(w)$-modules from A-modules.

Definition 3.6. Let $\{M(i) \mid i \in I\}$ be a set of A-modules. Given any $\boldsymbol{\lambda} = (\lambda^i)_{i\in I} \in \Lambda^I_w$, we construct an $A(w)$-module $M(\boldsymbol{\lambda})$ as follows:

$$M(\boldsymbol{\lambda}) := \mathrm{Ind}^w_{(|\lambda^i|)_{i\in I}}\left(\bigotimes_{i\in I} T^{\lambda^i}(M(i))\right).$$

In view of Lemma 3.1(3), the order in which the tensor product is taken is not important.

The fact that the module $M(\boldsymbol{\lambda})$ depends on the $M(i)$'s is not made explicit by our notation, but we believe there shouldn't be too much danger of confusion.

This is a natural and important construction. For example,

Proposition 3.7 (Macdonald [7, p. 204])**.** *Suppose that $R = k$ is a splitting field for A. Let $\{M(i) \mid i \in I\}$ be a complete set of nonisomorphic simple A-modules. Then $A(w)$ is a split k-algebra and $\{M(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_w\}$ is a complete set of nonisomorphic simple $A(w)$-modules.*

The following observation that will prove useful.

Lemma 3.8. *Let M be an A-module whose direct summands are $\{M(i) \mid i \in I\}$. Then the direct summands of the $A(w)$-module $\mathrm{Ind}^w_{(1^w)}(T^w(M))$ are $\{M(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_w\}$.*

Proof. By Lemma 3.1(2), $\mathrm{Ind}^w_{(1^w)}(T^w(M))$ is isomorphic to a direct sum of modules

$$\mathrm{Ind}^w_{(w_i)_{i\in I}}\left(\bigotimes_{i\in I} \mathrm{Ind}^{w_i}_{(1^{w_i})} T^{w_i}(M(i))\right),$$

where $(w_i)_{i\in I}$ is an I-tuple of nonnegative integers summing to w and each such tuple actually occurs. Each term $\mathrm{Ind}^{w_i}_{(1^{w_i})} T^{w_i}(M(i))$ is by Lemma 3.3(1) in turn isomorphic to a direct sum of $T^{\lambda^i}(M(i))$'s for $\lambda^i \vdash w_i$ where each such partition occurs. The statement follows.

Corollary 3.9. *Suppose that $\{M(i) \mid i \in I\}$ is a set of projective A-modules. Then $M(\boldsymbol{\lambda})$ is projective for all $\boldsymbol{\lambda} \in \Lambda^I_w$.*

Proof. It suffices to show the Corollary for the case where $\{M(i) \mid i \in I\}$ is a complete set of indecomposable projective A-modules. In this case, we apply Lemma 3.8 with $M = A$, and conclude that $M(\boldsymbol{\lambda})$ is a direct summand of $\mathrm{Ind}^w_{(1^w)}(T^w(A)) = A(w)$, and hence is projective.

Remark.

(1) All of the constructions in this section are well-behaved under base change. Let $R_1 \to R_2$ be a ring homomorphism. For example R_2 can be the fraction field or residue field of R_1, when R_1 is a discrete valuation ring. Given an R_1-algebra A_1 we set $A_2 = R_2 \otimes_{R_1} A_1$. Firstly we have an obvious isomorphism of R_2-algebras
$$R_2 \otimes_{R_1} A_1(w) \cong A_2(w).$$
If $\{M_1(i) \mid i \in I\}$ is a set of A_1-modules and $\boldsymbol{\lambda} \in \Lambda^I_w$, then we can construct, as above, an $A_1(w)$-module $M_1(\boldsymbol{\lambda})$ as well as an $A_2(w)$-module $M_2(\boldsymbol{\lambda})$ (from the modules $M_2(i) = R_2 \otimes_{R_1} M_1(i)$). There is an isomorphism of A_2-modules
$$R_2 \otimes_{R_1} M_1(\boldsymbol{\lambda}) \cong M_2(\boldsymbol{\lambda}).$$

(2) The constructions in this section are functorial in an obvious way. In particular, given sets $\{M(i) \mid i \in I\}$ and $\{N(i) \mid i \in I\}$ of A-modules, along with homomorphisms $\{\phi(i) : M(i) \to N(i) \mid i \in I\}$, we get canonically defined homomorphisms $\phi(\boldsymbol{\lambda}) : M(\boldsymbol{\lambda}) \to N(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \Lambda^I_w$.

(3) There are obvious analogous versions of all the constructions in this section for right modules. In particular given a set of right A-modules $\{M'(i) \mid i \in I\}$ we can construct for any $\boldsymbol{\lambda} \in \Lambda^I_w$ a right

$A(w)$-modules $M'(\boldsymbol{\lambda})$. Moreover if $\{M(i) \mid i \in I\}$ is a set of left A-modules and $M(i)^\vee \cong M'(i)$, then a straightforward albeit tedious argument shows that $M(\boldsymbol{\lambda})^\vee \cong M'(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \Lambda_w^I$.

4. Filtrations

In this section we investigate how filtrations of modules behave with respect to the constructions described in the previous section.

We begin with a two lemmas, which handle the most basic case.

Lemma 4.1. *Let A and B be R-algebras. Let M be an A-module having a filtration*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_m \supseteq M_{m+1} = 0,$$

and N be a B-module having a filtration

$$N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_n \supseteq N_{n+1} = 0.$$

Assume that the subquotients M_i/M_{i+1} and N_j/N_{j+1} are all R-free. Then the $A \otimes B$-module $M \otimes N$ is filtered by

$$\left\{ \frac{M_i}{M_{i+1}} \otimes \frac{N_j}{N_{j+1}} \mid 0 \leq i \leq m,\ 0 \leq j \leq n \right\}.$$

Proof. Let $V_{i,j} = M_{i+1} \otimes N + M_i \otimes N_j$. Then

$$M \otimes N = V_{0,0} \supseteq V_{0,1} \supseteq \cdots \supseteq V_{0,n} \supseteq V_{1,0} \supseteq V_{1,1} \supseteq \cdots \supseteq V_{m,n} \supseteq 0$$

is a filtration of $M \otimes N$, and $\frac{V_{i,j}}{V_{i,j+1}} \cong \frac{M_i}{M_{i+1}} \otimes \frac{N_j}{N_{j+1}}$ if $j < n$ and $\frac{V_{i,n}}{V_{i+1,0}} \cong \frac{M_i}{M_{i+1}} \otimes N_n$.

Lemma 4.2. *Let M be an A-module having a filtration*

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_m \supseteq M_{m+1} = 0$$

such that each subquotient M_i/M_{i+1} is R-free. Then the $A(w)$-module $T^{(w)}(M)$ is filtered by the set

$$\left\{ \mathrm{Ind}_{\mathsf{w}}^{w} \left(\bigotimes_{s=0}^{m} T^{(w_s)} \left(\frac{M_s}{M_{s+1}} \right) \right) \mid \mathsf{w} = (w_0, w_1, \ldots, w_m) \vDash w \right\}.$$

Proof. We can choose an R-basis $\mathcal{B}$ for M along with a function $g : \mathcal{B} \to \{0, \dots, m\}$ so that for $s \in \{0, \dots, m\}$, the subset $\{b \in \mathcal{B} \mid g(b) \geq s\}$ is a basis for M_s. Hence if $b_0 \in \mathcal{B}$ and $a \in A$, then $ab_0 \in \operatorname{span}_R\{b \in \mathcal{B} \mid g(b) \geq g(b_0)\}$. It is clear that $\{b_1 \otimes \cdots \otimes b_w \mid b_1, \dots, b_w \in \mathcal{B}\}$ is an R-basis of $T^{(w)}(M)$. Define the *weight* of $b_1 \otimes \cdots \otimes b_w$ to be the composition $\mathsf{w} = (w_0, w_1, \dots, w_m) \vDash w$ where $w_s = |\{i \mid g(b_i) = s\}|$ for all s. Note that the set of basis elements with a given weight is invariant under the action of $\mathfrak{S}_w$. For each nonnegative integer n, let Z_n be the R-span of

$$\left\{ b_1 \otimes \cdots \otimes b_w \mid b_i \in \mathcal{B},\ \sum_{i=1}^{w} g(b_i) \geq n \right\}.$$

Then Z_n is an $A(w)$-submodule of $T^{(w)}(M)$, and is in fact equal to

$$\sum_{(n_1,\dots,n_w) \vDash n} \left(\bigotimes_{i=1}^{w} M_{n_i} \right).$$

We have a filtration

$$T^{(w)}(M) = Z_0 \supseteq Z_1 \supseteq \cdots,$$

and Z_n/Z_{n+1} has a basis $\{b_1 \otimes \cdots \otimes b_w + Z_{n+1} \mid b_i \in \mathcal{B}, \sum_{i=1}^{w} g(b_i) = n\}$. For each $\mathsf{w} = (w_0, \dots, w_m) \vDash w$ such that $\sum_{s=0}^{m} s w_s = n$, let V_{w} be the R-span in Z_n/Z_{n+1} of the basis elements with weight w. Then V_{w} is an $A(w)$-submodule of Z_n/Z_{n+1}, and

$$Z_n/Z_{n+1} = \bigoplus_{\substack{\mathsf{w}=(w_0,\dots,w_m)\vDash w \\ \sum_{s=0}^{m} s w_s = n}} V_{\mathsf{w}}.$$

Thus $T^{(w)}(M)$ is filtered by the set $\{V_{\mathsf{w}} \mid \mathsf{w} = (w_0, \dots, w_m) \vDash w\}$. We now provide a description of V_{w}. Let $\sum_{s=0}^{m} s w_s = n$, and consider the R-submodule V_0 of Z_n/Z_{n+1} spanned by elements of the form $b_1 \otimes \cdots \otimes b_w + Z_{n+1}$ with $g(b_1) = \cdots = g(b_{w_0}) = 0$, $g(b_{w_0+1}) = \cdots = g(b_{w_0+w_1}) = 1$, etc. Note that V_0 is an $A(\mathsf{w})$-submodule of Z_n/Z_{n+1} isomorphic to

$$\bigotimes_{s=0}^{m} T^{(w_s)}\left(\frac{M_s}{M_{s+1}} \right).$$

As an R-module, $V_{\mathsf{w}} = \bigoplus_{\sigma \in \mathfrak{S}_w/\mathfrak{S}_{\mathsf{w}}} (1 \otimes \sigma) V_0$. As such, $V_{\mathsf{w}} = \operatorname{Ind}_{A(\mathsf{w})}^{A(w)} V_0 \cong \operatorname{Ind}_{A(\mathsf{w})}^{A(w)} \left(\bigotimes_{s=0}^{m} T^{(w_s)} \left(\frac{M_s}{M_{s+1}} \right) \right)$.

When R is a field and the filtration of M is an refinement of the radical filtration, we are able to obtain a formula for the radical layers of $T^{(w)}(M)$.

Lemma 4.3. *Suppose that $R = k$ is a field, and let M be an A-module having a filtration as in Lemma 4.2.*

(1) *Suppose $M_s = \operatorname{rad}^s(M)$. Then*

$$\frac{\operatorname{rad}^n(T^{(w)}(M))}{\operatorname{rad}^{n+1}(T^{(w)}(M))} \cong \bigoplus_{\mathsf{w}} \operatorname{Ind}_{\mathsf{w}}^{w}\left(\bigotimes_{s=0}^{m} T^{(w_s)}\left(\frac{\operatorname{rad}^s(M)}{\operatorname{rad}^{s+1}(M)}\right)\right),$$

where $\mathsf{w} = (w_0, w_1, \ldots, w_m)$ runs over all compositions of w such that $\sum_{s=0}^{m} s w_s = n$.

(2) *Suppose that the given filtration of M is a refinement of the radical filtration, so that for $s = 0, 1, \ldots, m$, we have $\operatorname{rad}^{l_s}(M) \supseteq M_s \supsetneqq \operatorname{rad}^{l_s+1}(M)$ for a unique l_s. Then*

$$\frac{\operatorname{rad}^n(T^{(w)}(M))}{\operatorname{rad}^{n+1}(T^{(w)}(M))} \cong \bigoplus_{\mathsf{w}} \operatorname{Ind}_{\mathsf{w}}^{w}\left(\bigotimes_{s=0}^{m} T^{(w_s)}\left(\frac{M_s}{M_{s+1}}\right)\right),$$

where $\mathsf{w} = (w_0, w_1, \ldots, w_m)$ runs over all compositions of w such that $\sum_{s=0}^{m} w_s l_s = n$.

Proof. For part (1), keeping the notations used in the proof of Lemma 4.2, it suffices to show that $Z_n = \operatorname{rad}^n(T^{(w)}(M))$. Using Lemma 3.4(1), we see that

$$\operatorname{rad}^n(T^w(A)) = \sum_{(n_1,\ldots,n_w)\vDash n} \left(\bigotimes_{i=1}^{w} \operatorname{rad}^{n_i}(A)\right).$$

Now

$$\begin{aligned} Z_n &= \sum_{(n_1,\ldots,n_w)\vDash n} \left(\bigotimes_{i=1}^{w} \operatorname{rad}^{n_i}(M)\right) \\ &= \sum_{(n_1,\ldots,n_w)\vDash n} \left(\bigotimes_{i=1}^{w} \operatorname{rad}^{n_i}(A)\right)(T^w(M)) \\ &= \operatorname{rad}^n(T^w(A))(T^w(M)) = \operatorname{rad}^n(T^{(w)}(M)), \end{aligned}$$

the last equality by Lemma 3.5(2).

For part (2), let $M_r = \operatorname{rad}^s(M)$ and $M_t = \operatorname{rad}^{s+1}(M)$, and thus $l_i = s$ for all $r \le i < t$. Using Lemma 4.2, we know that $T^{(v_s)}(M_r/M_t)$ is filtered by the set

$$\left\{ \mathrm{Ind}^{v_s}\left(\bigotimes_{i=r}^{t-1} T^{(w_i)}\left(\frac{M_i}{M_{i+1}}\right)\right) \mid (w_r, w_{r+1}, \ldots, w_{t-1}) \vDash v_s \right\}.$$

The statement thus follows from part (1) and Lemma 3.1(2).

We now introduce the main result on filtrations. Let $\{M(i) \mid i \in I\}$ and $\{N(j) \mid j \in J\}$ be sets of R-free A-modules and suppose that each $M(i)$ has a filtration whose subquotients are isomorphic to $N(j)$'s. We will construct filtrations of the $A(w)$-modules $M(\boldsymbol{\lambda})$ ($\boldsymbol{\lambda} \in \Lambda_w^I$) in which the subquotients are isomorphic to $N(\boldsymbol{\mu})$'s ($\boldsymbol{\mu} \in \Lambda_w^J$) (see Definition 3.6). We shall keep track of multiplicities and give additional information on the radical filtrations of the $M(\boldsymbol{\lambda})$'s when the original filtrations are refinements of radical filtrations. We shall assume for the sake of simplicity that the $N(j)$'s are pairwise nonisomorphic.

We write down the given filtrations

$$(*) \qquad M(i) = M(i,0) \supseteq M(i,1) \supseteq \cdots \supseteq M(i, m_i + 1) = 0,$$

set $K = \{(i,s) \in I \times \mathbb{Z} \mid 0 \leq s \leq m_i\}$, and let

$$F(i,s) = \frac{M(i,s)}{M(i,s+1)}$$

for each $(i,s) \in K$. By assumption each $F(i,s)$ is R-free and isomorphic to a unique $N(j)$. Let $K_j = \{(i,s) \in K \mid F(i,s) \cong N(j)\}$.

Proposition 4.4. *Let $\boldsymbol{\lambda} \in \Lambda_w^I$.*

(1) *The $A(w)$-module $M(\boldsymbol{\lambda})$ has a filtration with subquotients isomorphic to $N(\boldsymbol{\mu})$'s ($\boldsymbol{\mu} \in \Lambda_w^J$) in which $N(\boldsymbol{\mu})$ appears*

$$(**) \qquad \sum_{\boldsymbol{\rho}} \left(\prod_{i \in I} c(\lambda^i; \rho^{i0}, \ldots, \rho^{im_i}) \cdot \prod_{j \in J} c(\mu^j; (\rho^{is} \mid (i,s) \in K_j)) \right)$$

times, with the sum taken over all $\boldsymbol{\rho} = (\rho^{is}) \in \Lambda_w^K$.

(2) *Suppose that $R = k$ is a splitting field for A and the filtrations in $(*)$ are composition series refining the radical filtrations (thus the $N(j)$'s are simple). For $(i,s) \in K$, we define l_{is} by*

$$\mathrm{rad}^{l_{is}}(M(i)) \supseteq M(i,s) \supsetneq \mathrm{rad}^{l_{is}+1}(M(i)).$$

Then the multiplicity of the simple $A(w)$-module $N(\boldsymbol{\mu})$ in the r-th radical layer of $M(\boldsymbol{\lambda})$ is given by $(**)$, with the sum taken over all $\boldsymbol{\rho} = (\rho^{is}) \in \Lambda_w^K$ satisfying $\sum_{(i,s)\in K} |\rho^{is}| l_{is} = r$.

Proof. We will make repeated use of Lemmas 3.2 and 3.3 without comment.

Let $\mathsf{w} = (w_i \mid i \in I) \vDash w$. Firstly, by Lemma 4.2, we know that for each $i \in I$, the $A(w_i)$-module $T^{(w_i)}(M(i))$ is filtered by the set

$$\left\{ \mathrm{Ind}_{\mathsf{v}_i}^{w_i} \left(\bigotimes_{s=0}^{m_i} T^{(v_{is})}(F(i,s)) \right) \mid \mathsf{v}_i = (v_{i0}, \dots, v_{im_i}) \vDash w_i \right\}.$$

If $\lambda^i \vdash w_i$, then

$$\begin{aligned}
&\mathrm{Ind}_{\mathsf{v}_i}^{w_i} \left(\bigotimes_{s=0}^{m_i} T^{(v_{is})}(F(i,s)) \right) \oslash S^{\lambda^i} \\
&\cong \mathrm{Ind}_{\mathsf{v}_i}^{w_i} \left(\left(\bigotimes_{s=0}^{m_i} T^{(v_{is})}(F(i,s)) \right) \oslash \mathrm{Res}_{\mathsf{v}_i}^{w_i}(S^{\lambda^i}) \right) \\
&\cong \mathrm{Ind}_{\mathsf{v}_i}^{w_i} \left(\left(\bigotimes_{s=0}^{m_i} T^{(v_{is})}(F(i,s)) \right) \oslash \bigoplus_{(\rho^{is} \vdash v_{is})_s} c(\lambda^i; \rho^{i0}, \dots, \rho^{im_i}) \left(\bigotimes_{s=0}^{m_i} S^{\rho^{is}} \right) \right) \\
&\cong \bigoplus_{(\rho^{is} \vdash v_{is})_s} c(\lambda^i; \rho^{i0}, \dots, \rho^{im_i}) \, \mathrm{Ind}_{\mathsf{v}_i}^{w_i} \left(\bigotimes_{s=0}^{m_i} T^{\rho^{is}}(F(i,s)) \right).
\end{aligned}$$

Thus the $A(w_i)$-module $T^{\lambda^i}(M(i)) = T^{(w_i)}(M(i)) \oslash S^{\lambda^i}$ is filtered by the set

$$\left\{ c(\lambda^i; \rho^{i0}, \dots, \rho^{im_i}) \, \mathrm{Ind}_{\mathsf{v}_i}^{w_i} \left(\bigotimes_{s=0}^{m_i} T^{\rho^{is}}(F(i,s)) \right) \mid (\rho^{is} \vdash v_{is})_s, \ \mathsf{v}_i = (v_{i0}, \dots, v_{im_i}) \vDash w_i \right\}.$$

It follows from Lemma 4.1 that $M(\boldsymbol{\lambda}) = \mathrm{Ind}_{\mathsf{w}}^{w} \left(\bigotimes_{i\in I} T^{\lambda_i}(M(i)) \right)$ is filtered by the set (see Definition 3.6)

$$\left\{ \left(\prod_{i\in I} c(\lambda^i; \rho^{i0}, \dots, \rho^{im_i}) \right) F(\boldsymbol{\rho}) \mid \boldsymbol{\rho} = (\rho^{is}) \in \Lambda_w^K \right\}.$$

Given any $\boldsymbol{\rho} = (\rho^{is}) \in \Lambda_w^K$, let $v_{is} = |\rho^{is}|$, $w_j = \sum_{(i,s)\in K_j} v_{is}$, $\mathsf{w} = (w_j)_{j\in J}$, $\mathsf{v} = (v_{is})_{(i,s)\in K}$, and $\mathsf{v}_j = (v_{is})_{(i,s)\in K_j}$. Then

$$F(\boldsymbol{\rho}) = \mathrm{Ind}_{\mathsf{v}}^{w} \left(\bigotimes_{(i,s)\in K} T^{\rho^{is}}(F(i,s)) \right)$$

$$\begin{aligned}
&\cong \mathrm{Ind}_{\mathsf{w}}^{w}\left(\bigotimes_{j\in J}\mathrm{Ind}_{\mathsf{v}_j}^{w_j}\left(\bigotimes_{(i,s)\in K_j} T^{\rho^{is}}(N(j))\right)\right)\\
&\cong \mathrm{Ind}_{\mathsf{w}}^{w}\left(\bigotimes_{j\in J}\mathrm{Ind}_{\mathsf{v}_j}^{w_j}\left(\left(\mathrm{Res}_{\mathsf{v}_j}^{w_j} T^{(w_j)}(N(j))\right)\oslash \bigotimes_{(i,s)\in K_j} S^{\rho^{is}}\right)\right)\\
&\cong \mathrm{Ind}_{\mathsf{w}}^{w}\left(\bigotimes_{j\in J}\left(T^{(w_j)}(N(j))\oslash \mathrm{Ind}_{\mathsf{v}_j}^{w_j}\left(\bigotimes_{(i,s)\in K_j} S^{\rho^{is}}\right)\right)\right)\\
&= \mathrm{Ind}_{\mathsf{w}}^{w}\left(\bigotimes_{j\in J}\left((T^{(w_j)}(N(j))\oslash \left(\bigoplus_{\mu^j\vdash w_j} c(\mu^j;(\rho^{is})_{(i,s)\in K_j})S^{\mu^j}\right)\right)\right)\\
&= \bigoplus_{\boldsymbol{\mu}}\left(\prod_{j\in J} c(\mu^j;(\rho^{is})_{(i,s)\in K_j})\right)N(\boldsymbol{\mu}),
\end{aligned}$$

where the sum runs over all J-tuples $\boldsymbol{\mu}=(\mu^j \mid j\in J)$ of partitions.

Putting this together with our calculations above we obtain the first statement of the Proposition.

Now we assume that $R=k$ is a splitting field for A and that the filtrations of the $M(i)$ are composition series refining the radical filtrations: for each $(i,s)\in K$ we have $\mathrm{rad}^{l_{is}}(M(i))\supseteq M(i,s)\supsetneq \mathrm{rad}^{l_{is}+1}(M(i))$ for some l_{is}.

Firstly, by Lemma 4.3(2) we know that for each $i\in I$, the r-th radical layer of the $A(w_i)$-module $T^{(w_i)}(M(i))$ is isomorphic to

$$\bigoplus_{\mathsf{v}_i}\mathrm{Ind}_{\mathsf{v}_i}^{w_i}\left(\bigotimes_{s=0}^{m_i} T^{(v_{is})}(F(i,s))\right),$$

where the sum is over $\mathsf{v}_i=(v_{i0},\dots,v_{im_i})\vDash w_i$ such that $\sum_{s=0}^{m_i} v_{is}l_{is}=r$. Thus by Lemmas 3.5(3), 3.2(4) and 2.1(2), the r-th radical layer of the $A(w_i)$-module $T^{\lambda^i}(M(i))=T^{(w_i)}(M(i))\oslash S^{\lambda^i}$ is isomorphic to

$$\bigoplus_{\mathsf{v}_i}\left(\bigoplus_{(\rho^{is}\vdash v_{is})_s}\mathrm{Ind}_{\mathsf{v}_i}^{w_i}\left(\bigotimes_{s=0}^{m_i} c(\lambda^i;\rho^{i0},\dots,\rho^{im_i})\left(T^{(v_{is})}(F(i,s))\oslash S^{\rho^{is}}\right)\right)\right),$$

where the outer sum is again over $\mathsf{v}_i=(v_{i0},\dots,v_{im_i})\vDash w_i$ such that $\sum_{s=0}^{m_i} v_{is}l_{is}=r$.

Hence, using Lemma 3.5(4) and the argument in the proof above, the r-th radical layer of $M(\boldsymbol{\lambda})$ is a direct sum of $F(\boldsymbol{\rho})$'s for $\boldsymbol{\rho}=(\rho^{is})\in\Lambda_w^K$ satisfying $\sum_{(i,s)\in K}|\rho^{is}|l_{is}=r$, and $F(\boldsymbol{\rho})$ appears with multiplicity $\prod_{i\in I} c(\lambda^i;\rho^{i0},\dots,\rho^{im_i})$.

Finally we express each $F(\boldsymbol{\rho})$ as a direct sum of simple modules $N(\boldsymbol{\mu})$ as above and obtain the second statement in the Proposition.

As an easy application of our results, we have the following:

Lemma 4.5. *Suppose $R = k$ is a splitting field for A. Let $\{M(i) \mid i \in I\}$ be a set of A-modules, and suppose that for each $i \in I$, $M(i)$ has a simple head $L(i)$, and $L(i) \not\cong L(j)$ if $i \neq j$. Then $M(\boldsymbol{\lambda})$ has a simple head isomorphic to $L(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \Lambda^I_w$.*

Proof. Let $\{N(j) \mid j \in J\}$ be a complete set of simple A-modules. We may assume that $J \supseteq I$ and $N(i) = L(i)$ for all $i \in I$. We then apply Proposition 4.4(2). Note that $l_{is} = 0$ if and only if $s = 0$, and $(i, 0) \in K_i$. Thus $\sum_{(i,s)\in K} |\rho^{is}| l_{is} = 0$ implies that $\rho^{is} = \emptyset$ for all $s \geq 1$, so that $\lambda^i = \rho^{i0} = \mu^i$.

Proposition 4.6. *Let $R = k$ be a splitting field for A, and let $\{L(i) \mid i \in I\}$ be the simple A-modules and $\{P(i) \mid i \in I\}$ their projective covers.*

(1) *For each $\boldsymbol{\lambda} \in \Lambda^I_w$, $P(\boldsymbol{\lambda})$ is the projective cover of $L(\boldsymbol{\lambda})$.*

(2) *(Ext^1-quiver)*

  (a) *For $\boldsymbol{\lambda} \in \Lambda^I_w$, we have*

$$\dim_k \mathrm{Ext}^1_{A(w)}(L(\boldsymbol{\lambda}), L(\boldsymbol{\lambda})) = \sum_{i\in I} p(\lambda^i) \dim_k \mathrm{Ext}^1_A(L(i), L(i)),$$

  where $p(\lambda^i)$ is the number of distinct parts of λ^i.

  (b) *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^I_w$, $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$, we have $\mathrm{Ext}^1_{A(w)}(L(\boldsymbol{\lambda}), L(\boldsymbol{\mu})) = 0$ unless either*

  - *there exists $j \in I$, such that $\lambda^i = \mu^i$ for all $i \in I$, $i \neq j$,*
  - *there exists $\nu \in \Lambda$ such that both λ^j and μ^j are obtained from ν by adding one node,*
  - $\mathrm{Ext}^1_A(L(j), L(j)) \neq 0$,

  in which case

$$\dim_k \mathrm{Ext}^1_{A(w)}(L(\boldsymbol{\lambda}), L(\boldsymbol{\mu})) = \dim_k \mathrm{Ext}^1_A(L(j), L(j));$$

  or

- *there exist* $j, j' \in I$, $j \neq j'$, *such that* $\lambda^i = \mu^i$ *for all* $i \in I$, $i \neq j$, $i \neq j'$
- μ^j *is obtained from* λ^j *by adding one node*
- $\mu^{j'}$ *is obtained from* $\lambda^{j'}$ *by removing one node*
- $\mathrm{Ext}^1_A(L(j), L(j')) \neq 0$,

in which case

$$\dim_k \mathrm{Ext}^1_{A(w)}(L(\boldsymbol{\lambda}), L(\boldsymbol{\mu})) = \dim_k \mathrm{Ext}^1_A(L(j), L(j')).$$

Proof. The first statement follows from Lemma 4.5 and Corollary 3.9. The second statement follows from Proposition 4.4(2) using the fact that, in the notation of the statement of that Proposition, $\sum_{(i,s)\in K} |\rho^{is}| l_{is} = 1$ implies there exists a unique $(i, s) \in K$ with $s > 0$ such that $\rho^{is} \neq \emptyset$; furthermore, $\rho^{is} = (1)$ and $l_{is} = 1$.

If we are given a unitriangular system of filtrations of A-modules, then the resulting filtration of $A(w)$-modules is also unitriangular. More precisely,

Proposition 4.7. *Let* $\{M(i)\}$ *and* $\{N(i)\}$ *be two sets of* A*-modules indexed by a common partially ordered set* $(I, \geq)$*, and assume that each* $N(i)$ *is* R*-free. Suppose that for each* $i \in I$, $M(i)$ *has a filtration such that every subquotient is isomorphic to* $N(j)$ *for some* $j \leq i$*, and* $N(i)$ *occurs exactly once. Then each* $\boldsymbol{\lambda} \in \Lambda^I_w$, $M(\boldsymbol{\lambda})$ *has a filtration such that every subquotient is isomorphic to* $N(\boldsymbol{\mu})$ *for some* $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$*, and* $N(\boldsymbol{\lambda})$ *occurs exactly once.*

Proof. Our hypothesis implies that $(i, s) \in K_j$ only if $i \geq j$, so that $\bigcup_{j\geq t} K_j \subseteq \{(i, s) \in K \mid i \geq t\}$. Now, if a summand of $(**)$ in Proposition 4.4(1) is non-zero, then $|\mu^j| = \sum_{(i,s)\in K_j} |\rho^{is}|$ and $|\lambda^i| = \sum_{s=0}^{m_i} |\rho^{is}|$, so that

$$\sum_{j\geq t} |\mu^j| = \sum_{j\geq t} \sum_{(i,s)\in K_j} |\rho^{is}| \leq \sum_{i\geq t} \sum_{s=0}^{m_t} |\rho^{is}| = \sum_{i\geq t} |\lambda^i|,$$

i.e, $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$.

Now suppose $\boldsymbol{\mu} = \boldsymbol{\lambda}$. Then the above inequality is an equality for all $t \in I$. Let $I' = \{i \in I \mid \lambda^i \neq \emptyset\}$. Then I' is finite, and for all $i \in I \setminus I'$, $\rho^{is} = \emptyset$. We complete the proof by showing by induction that $\lambda^i = \rho^{is_i} = \mu^i$

for all $i \in I'$, where F_{is_i} is the unique subquotient of $M(i)$ isomorphic to $N(i)$.

Let t be a maximal member of I'. Then if $t' > t$, we have $\sum_{j \geq t'} |\mu^j| \leq \sum_{i \geq t'} |\lambda^i| = 0$; in particular, $\mu^{t'} = \emptyset$. Thus, $|\lambda^t| = \sum_{i \geq t} |\lambda^i| = \sum_{j \geq t} |\mu_j| = |\mu^t|$, which also shows that $\rho^{ts} \neq \emptyset$ if and only if $s = s_t$. This further yields $\lambda^t = \rho^{ts_t} = \mu^t$. The inductive step is similar.

5. The eAe construction.

Let $e \in A$ be an idempotent. Then eAe is a subalgebra of A (with identity element e), and we have an exact functor

$$f : A\text{-mod } \to eAe\text{-mod}$$

given by $f(M) = eM$ on objects and taking a homomorphism $M \to N$ to the restriction $eM \to eN$. Define

$$e_w = e^{\otimes w} \otimes 1 \in T^w(A) \otimes R\mathfrak{S}_w = A(w),$$

an idempotent in $A(w)$. Then it's easy to see that

$$e_w A(w) e_w = (eAe)(w)$$

and with this identification we have an exact functor

$$f_w : A(w)\text{-mod} \to (eAe)(w)\text{-mod}$$

defined in a similar manner as f. This functor has good properties with respect to the constructions studied in section 3:

Proposition 5.1.

(1) *If M is an A-module then*

$$f_w(M^{(w)}) = (f(M))^{(w)}.$$

(2) *If V is an $A(w)$-module and X an $R\mathfrak{S}_w$-module then*

$$f_w(V \oslash X) = f_w(V) \oslash X.$$

(3) *If $w = w_1 + w_2$, V_1 is an $A(w_1)$-module, and V_2 is an $A(w_2)$-module, then*

$$f_w(\mathrm{Ind}^w(V_1 \otimes V_2)) = \mathrm{Ind}^w(f_{w_1}(V_1) \otimes f_{w_2}(V_2)).$$

(4) *If $\{M(i) \mid i \in I\}$ is a collection of A-modules and we set $N(i) = f(M(i))$, then for any $\boldsymbol{\lambda} \in \Lambda_w^I$ we have*

$$f_w(M(\boldsymbol{\lambda})) = N(\boldsymbol{\lambda}).$$

Proof. These all follow directly from the definitions.

In some cases of interest (for example if A is a Schur algebra and f is the Schur functor [4, §6]) $\mathrm{End}_{eAe}(f(A)) \cong A$, where the isomorphism is given by right multiplication of A on $f(A) = eA$. This property passes to wreath products:

Proposition 5.2. *Suppose that the homomorphism*

$$A \to \mathrm{End}_{eAe}(f(A))$$

given by right multiplication of A on $f(A) = eA$ is an isomorphism. Then the homomorphism

$$A(w) \to \mathrm{End}_{(eAe)(w)}(f_w(A(w)))$$

given by right multiplication of $A(w)$ on $f_w(A(w)) = e_w A(w)$ is also an isomorphism.

Proof. First of all, the homomorphism

$$T^w(A) \to \mathrm{End}_{T^w(eAe)}(T^w(eA))$$

given by right multiplication of $T^w(A)$ on $T^w(eA)$ is an isomorphism, by (10.37) of [3] (which states the result for algebras over fields, but the same proof works for any algebra over a commutative ring R as long as the modules are free over R.)

Next, note that $f_w(A(w)) \cong T^{(w)}(eA) \oslash R\mathfrak{S}_w = \mathrm{Ind}^w_{(1^w)}(T^w(eA))$ and right multiplication by $\alpha \otimes \tau$ corresponds via the isomorphisms (see Lemma

3.2(2) and Lemma 3.1(1))

$$\begin{aligned}\mathrm{End}_{(eAe)(w)}(f_w(A(w))) &\cong \mathrm{Hom}_{(eAe)(w)}(\mathrm{Ind}^w_{(1^w)}(T^w(eA)), T^{(w)}(eA) \oslash R\mathfrak{S}_w)\\ &\cong \mathrm{Hom}_{T^w(eAe)}(T^w(eA), T^w(eA) \oslash \mathrm{Res}^w_{(1^w)} R\mathfrak{S}_w)\\ &\cong \mathrm{Hom}_{T^w(eAe)}(T^w(eA), \oplus_{\sigma\in\mathfrak{S}_w}(1\otimes\sigma)\otimes T^w(eA))\end{aligned}$$

to the homomorphism sending $m \in T^w(eA)$ to $(1\otimes\tau)\otimes m(\alpha\tau)$.

6. Quasihereditary algebras

Let A be a finite-dimensional algebra over a field k with simple modules $\{L(i) \mid i \in I\}$ indexed by a partially ordered set $(I, >)$. Recall (or see [2]) that a finite-dimensional k-algebra A is quasihereditary (with respect to $>$) if there exist A-modules $\{\Delta(i) \mid i \in I\}$ such that

(1) $\Delta(i)/\operatorname{rad}(\Delta(i)) \cong L(i)$ and every composition factor of $\operatorname{rad}(\Delta(i))$ is isomorphic to $L(j)$ for some $j < i$.

(2) $P(i)$ has a filtration

$$P(i) = P(i)_0 \supseteq P(i)_1 \supseteq \cdots \supseteq P(i)_{l+1} = 0$$

such that $P(i)_0/P(i)_1 \cong \Delta(i)$ and such that for each $\gamma \in \{1, 2, \ldots, l\}$, we have $P(i)_\gamma/P(i)_{\gamma+1} \cong \Delta(j)$ for some $j > i$.

If A is quasihereditary then the $\Delta(i)$'s are characterised up to isomorphism by properties (1) and (2), and are called the standard modules of A.

Now suppose that A is quasihereditary and split over k. Then $\{L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_w\}$ are the $A(w)$-simple modules, and by Proposition 4.6(1), $P(\boldsymbol{\lambda})$ is the projective cover of $L(\boldsymbol{\lambda})$. By Lemma 4.5, $\Delta(\boldsymbol{\lambda})$ has simple head isomorphic to $L(\boldsymbol{\lambda})$ and by Proposition 4.7, every composition factor of $\operatorname{rad}(\Delta(\boldsymbol{\lambda}))$ is isomorphic to $L(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$. Furthermore, Proposition 4.7 also shows that $P(\boldsymbol{\lambda})$ has a filtration in which each subquotient is isomorphic to $\Delta(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \succeq \boldsymbol{\lambda}$ and $\Delta(\boldsymbol{\lambda})$ occurs exactly once. But since $\Delta(\boldsymbol{\lambda})$ is the only subquotient that has head isomorphic to $L(\boldsymbol{\lambda})$, this subquotient must occur at the top. Thus $A(w)$ is a quasihereditary algebra with standard modules $\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_w\}$, with respect to the partial order $\succeq$.

7. Example

Let k be a field and n an integer ≥ 2. Let A be the path algebra over k of the quiver

$$\underset{0}{\bullet} \underset{\delta_1}{\overset{\gamma_1}{\rightleftarrows}} \underset{1}{\bullet} \underset{\delta_2}{\overset{\gamma_2}{\rightleftarrows}} \underset{2}{\bullet} \quad \cdots \quad \underset{n-2}{\bullet} \underset{\delta_{n-1}}{\overset{\gamma_{n-1}}{\rightleftarrows}} \underset{n-1}{\bullet}$$

modulo the ideal generated by

$$\{\gamma_i\gamma_{i+1},\ \delta_{i+1}\delta_i,\ \delta_i\gamma_i - \gamma_{i+1}\delta_{i+1} \mid 1 \leq i \leq n-2\} \cup \{\delta_{n-1}\gamma_{n-1}\}.$$

Let $\mathcal{L}(i)$ be the simple A-module corresponding to the vertex i, and let $\mathcal{P}(i)$ be a projective cover of $\mathcal{L}(i)$. The radical layers of the $\mathcal{P}(i)$'s are as follows:

$$\mathcal{P}(0) = \begin{matrix} \mathcal{L}(0) \\ \mathcal{L}(1) \\ \mathcal{L}(0) \end{matrix}, \quad \mathcal{P}(i) = \begin{matrix} & \mathcal{L}(i) & \\ \mathcal{L}(i-1) & & \mathcal{L}(i+1) \\ & \mathcal{L}(i) & \end{matrix} \quad (1 \leq i \leq n-2), \quad \mathcal{P}(n-1) = \begin{matrix} \mathcal{L}(n-1) \\ \mathcal{L}(n-2) \end{matrix}.$$

Let $\Omega(0) = \mathcal{L}(0)$, and for $i \in I = \{0, \ldots, n-1\}$, let $\Omega(i)$ be a nonsplit extension $\mathcal{L}(i)$ by $\mathcal{L}(i-1)$. Then it's easy to check that A is a quasihereditary algebra with simple modules $\mathcal{L}(i)$ and standard modules $\Omega(i)$ indexed by I with the natural order.

This is an important example: it is well-known that A is the basic algebra of any weight 1 block of any q-Schur algebra over k for which n is the least positive integer such that $1 + q + \ldots + q^{n-1} = 0$ in k. (this can be deduced, for instance, from [9, p.126, rule 13].)

Now let w be a positive integer such that $w!$ is invertible in k. By the result of the previous section, the algebra $A(w)$ is quasihereditary with simple modules $\mathcal{L}(\boldsymbol{\lambda})$ and standard modules $\Omega(\boldsymbol{\lambda})$ indexed by $\boldsymbol{\lambda} \in \Lambda^I_w$. Define for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^I_w$ polynomials

$$\operatorname{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = \sum_{r\geq 0} [\operatorname{rad}^r \Omega(\boldsymbol{\lambda}) / \operatorname{rad}^{r+1} \Omega(\boldsymbol{\lambda}) : \mathcal{L}(\boldsymbol{\mu})] v^r,$$

$$\operatorname{rad}_{\mathcal{P},\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = \sum_{r\geq 0} [\operatorname{rad}^r \mathcal{P}(\boldsymbol{\lambda}) / \operatorname{rad}^{r+1} \mathcal{P}(\boldsymbol{\lambda}) : \mathcal{L}(\boldsymbol{\mu})] v^r.$$

Proposition 7.1. *We have for* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_w^I$,

$$\text{(1)} \qquad \mathrm{rad}_{\Omega,\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = v^{\delta(\boldsymbol{\lambda},\boldsymbol{\mu})} \sum_{\substack{\alpha^0,\dots,\alpha^n \\ \beta^0,\dots,\beta^{n-1}}} \prod_{j=0}^{n-1} c(\lambda^j;\alpha^j,\beta^j)c(\mu^j;\beta^j,\alpha^{j+1})$$

where $\alpha^0,\dots,\alpha^n,\beta^0,\dots,\beta^{n-1}$ *run through* Λ *and*

$$\delta(\boldsymbol{\lambda},\boldsymbol{\mu}) = \sum_{j=1}^{n-1} j(|\lambda^j| - |\mu^j|).$$

Moreover

$$\text{(2)} \qquad \mathrm{rad}_{\mathcal{P},\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = \sum_{\boldsymbol{\nu}\in\boldsymbol{\Lambda}_w^I} \mathrm{rad}_{\Omega,\boldsymbol{\nu},\boldsymbol{\lambda}}(v)\,\mathrm{rad}_{\Omega,\boldsymbol{\nu},\boldsymbol{\mu}}(v).$$

Note that for every nonzero term in the sum in (1) we must have

$$|\alpha^i| = \sum_{j=0}^{i-1} |\mu^j| - |\lambda^j|, \quad |\beta^i| = |\lambda^i| + \sum_{j=0}^{i-1} |\lambda^j| - |\mu^j|.$$

Formula (1) has been independently discovered by Miyachi in [10]; we are following Leclerc-Miyachi's presentation of the formula [6].

Proof. Formula (1) is a direct application of Proposition 4.4(2). To get formula (2) is a little harder: we express both sides in terms of Littlewood-Richardson coefficients using Proposition 4.4(2) and then use the following identity which is valid for $\sigma,\tilde{\sigma},\tau,\tilde{\tau}\in\Lambda$:

$$\sum_{\lambda} c(\lambda;\sigma,\tilde{\sigma})c(\lambda;\tau,\tilde{\tau}) = \sum_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}} c(\sigma;\alpha,\beta)c(\tilde{\sigma};\tilde{\alpha},\tilde{\beta})c(\tau;\alpha,\tilde{\beta})c(\tilde{\tau};\tilde{\alpha},\beta).$$

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